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LETTER TO THE EDITOR

Critical properties of an Ising model with infinite-range coupling

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Abstract. A proper lower bound approximation to the true free energy per site of an Ising model with infinite-range interaction is obtained by the Migdal–Kadanoff real space recursion formulae. The results for the critical behaviour are confirmed by the mean field analysis.

It is well known that the Migdal–Kadanoff (Migdal 1975, Kadanoff 1976) approximation and the mean field theory are respectively a lower and an upper bound to the true free energy of a spin model. In a recent paper (Baracca *et al* 1981) the Ising model and the Potts model have been analysed by a variational procedure, which combines these two approximations. One of the conclusions of the paper was that no phase transition was directly evident in the free energy of the models under analysis looking at the Migdal–Kadanoff lower bound. The aim of this note is to study the free energy of an Ising-like model with an infinite-range coupling whose action is

$$A[s]=k\sum_{\langle i,j\rangle} s_i s_j+\frac{\lambda}{N}\sum_{i,j=1}^{N} s_i s_j, \qquad s_i=\pm 1, \tag{1}$$

where $\langle i, j\rangle$ represents the ensemble of all the nearest-neighbour sites of a hypercubic lattice Λ composed of N sites. It is known that the Curie–Weiss model of ferromagnetism (Stanley 1971, Kac 1968), which is the second term of the RHS of (1), is exactly solvable by the mean field theory; but this is only an ingredient of our model, which contains also an Ising coupling k. The model can be interpreted as follows: a nearest-neighbour local interaction is superposed onto an infinite-range interaction triggered by a parameter λ which couples each variable s_i with a 'mean field' given by $N^{-1}\Sigma_i s_i$. In the disordered phase where $\Sigma_i s_i \sim \sqrt{N}$ this infinite-range coupling is negligible—at least for large N—while in the ordered phase where $\Sigma_i s_i \sim N$ the λ coupling plays a crucial role. Let us stress that when $i=j$ we obtain an additive constant λ, which has no influence on the evaluation of the physical quantities, and when i is nearest neighbour to j we get the contribution $2d\lambda N^{-1}$ to the k coupling. Consequently the action (1) should be written as follows:

$$A[s]=\tilde{k}\sum_{\langle i,j\rangle} s_i s_j+\frac{\lambda}{N}\sum_{\rangle i,j\langle} s_i s_j+\lambda \tag{1'}$$

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where $\tilde{k} = k + 2d\lambda N^{-1}$ and $)i, j($ represents the ensemble of all the pairs of sites beyond the nearest neighbours. In the actual calculations we shall use the action (1), since in the thermodynamic limit, i.e. $N \to \infty$, (1) and (1′) have the same critical behaviour. We shall prove that the Migdal–Kadanoff behaviour for the free energy shows directly a phase transition in the parameter space spanned by k and λ.

A direct application of the renormalisation group to the model (1) is not feasible, since the reduction of the degrees of freedom should be done simultaneously over all the interaction scales. The crucial point to overcome this obstacle is the application of a gaussian transformation of the partition function Z,

$$Z = \left(\frac{N}{4\pi\lambda}\right)^{1/2} \mathrm{Tr}_s \int_{-\infty}^{+\infty} \mathrm{d}h \exp\left(k \sum_{\langle i,j\rangle} s_i s_j + h \sum_i s_i - \frac{Nh^2}{4\lambda}\right). \qquad (2)$$

Since h is independent of the s_i, the order of the integrations can be inverted.

In what follows we shall consider only the case $\lambda > 0$, in which the RHS of (2) can be interpreted as the partition function of an Ising model in an external magnetic field with a gaussian distribution. The case $\lambda < 0$ would imply an imaginary magnetic field, which would complicate the Migdal–Kadanoff treatment.

The calculation of the partition function of the model (1) is performed in two steps: first we sum over the configurations of the s_i using the Midgal–Kadanoff decimation technique doubling the lattice spacing, and then we perform the integral over h. This calculation is done on a finite lattice Λ with periodic boundary conditions and composed of $N = 2^{ld}$ sites, where d is the dimension. We obtain the following recursion formulae for the couplings:

$$h' = h + \tfrac{1}{2}d \ln[\cosh(2^d k + h)/\cosh(2^d k - h)],$$
$$k' = \tfrac{1}{4} \ln[\cosh(2^d k + h)\cosh(2^d k - h)/\cosh^2 h]. \qquad (3)$$

The 'free energy' per site at fixed h is

$$F(k, \lambda; h) = -\frac{1}{N} \ln Z(h) = \frac{h^2}{4\lambda} - \sum_{n=1}^{l} \frac{u_{(n-1)}}{2^{nd}} + F_{\text{final}} \qquad (4)$$

where

$$Z(h) = \mathrm{Tr}_s \exp\left(k \sum_{\langle i,j\rangle} s_i s_j + h \sum_i s_i\right) \exp\left(-\frac{h^2 N}{4\lambda}\right)$$

and

$$u_{(n)} = \tfrac{1}{4}d \ln[\cosh(2^d k_n + h_n)\cosh(2^d k_n - h_n)\cosh^2 h_n] + (2^d - d - 1)\ln\cosh h_n.$$

The contribution of the last integration with periodic boundary conditions is

$$F_{\text{final}} = -(1/2^{ld})(k_l d + \ln\cosh h_l). \qquad (5)$$

Now we must integrate over h to obtain the expression of the free energy per site $\mathcal{F}(k, \lambda)$ of the model (1) defined as

$$-\mathcal{F}(k, \lambda) = \frac{1}{N} \ln \int_{-\infty}^{+\infty} \mathrm{d}h \exp[-NF(k, \lambda; h)]. \qquad (6)$$

This last integral can be evaluated by the saddle point method, which is a good approximation for N sufficiently large. We have verified that the function $F(k, \lambda; h)$

is symmetric in h, and that the stationary points are its minima. Therefore

$$\mathscr{F}(k, \lambda) = \min_{|h|<\infty} F(k, \lambda; h). \tag{7}$$

The neglected terms are $O(\ln N/N)$ coming from the saddle point approximation and from the normalisation of the gaussian transformation. We have studied numerically the behaviour of $\mathscr{F}(k, \lambda)$ in $d = 2$ and we have found evidence of a discontinuity of some derivatives of $\mathscr{F}(k, \lambda)$. For instance, if one looks at the shape of $\mathscr{F}$ as a function of λ at fixed k one gets a *strict* 'plateau' region (the value of the free energy is constant up to the twelfth significant figure) up to some 'critical' λ_c, after which $\mathscr{F}$ starts to decrease. This happens in a certain range of k. Now if we fix λ and vary k although we haven't any plateau region, a discontinuity is still present in one of the derivatives of $\mathscr{F}$ with respect to k at some k_c. These two procedures are consistent since they give the same critical line (k_c, λ_c). The origin of this 'non-analytic' behaviour of $\mathscr{F}$ is probably twofold: in part it comes from the recursion procedure applied many times (we have considered a lattice up to 2^{60} sites) and also from the saddle point approximation, which is valid for large N. We have interpreted this behaviour as the presence of phase transitions, which are pictured by a line in the plane (k, λ) (see

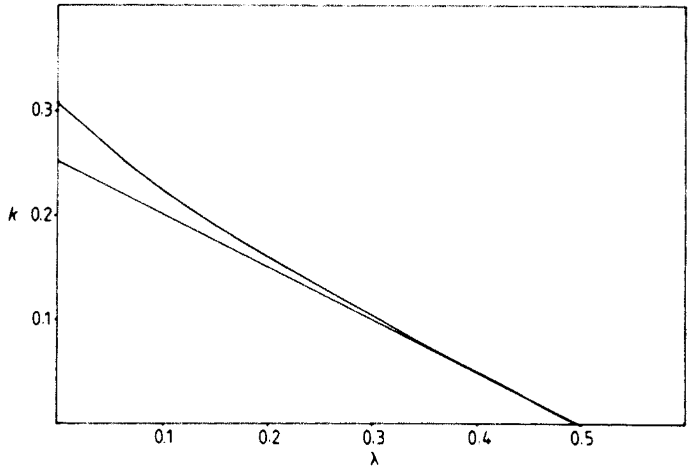

Figure 1. The critical line on the plane (k, λ) for the mean field (straight line) and for the Migdal–Kadanoff renormalisation technique (curved line).

figure 1). This line intersects the k axis at the Migdal–Kadanoff critical fixed point $(k_c = 0.305)$. As for the λ axis, on which the model is exactly solvable, we can obtain analytically the exact value of λ_c using formulae (3)–(7). In fact when $k = 0$ also $k' = 0$ and $h' = h$. Therefore (7) becomes

$$\mathscr{F}(0, \lambda) = \min_{|h|<\infty} \left(-3 \ln \cosh h \sum_{n=1}^{l} 2^{-nd} - \frac{1}{2^{ld}} \ln \cosh h + \frac{h^2}{4\lambda} \right). \tag{8}$$

Now we can identify the stationary points by direct calculations. The condition

$$\partial F(0, \lambda; h)/\partial h = 0 \tag{9}$$

gives a 'consistency' equation, similar to a molecular field equation:

$$\left(3\frac{1-2^{-ld}}{2^d-1}+\frac{1}{2^{ld}}\right)\tanh h-\frac{h}{2\lambda}=0. \tag{10}$$

The value of λ_c can be derived by the following condition on the second derivative:

$$\frac{\partial^2}{\partial h^2}F(0,\lambda;h)\bigg|_{h=0}=0. \tag{11}$$

One obtains in the limit $N\to\infty$ ($l\to\infty$) $\lambda_c=\frac{1}{2}$.

By numerical computations for $l=7$ we have found that the discontinuity in the second derivative of $\mathscr{F}$ near the critical points is much larger than the discontinuity in the first derivative. For instance, at $k=1.5$ around $\lambda_c=0.228$ the discontinuity in the second derivative is five orders of magnitude larger than in the first derivative, which is $\sim 10^{-3}$. Therefore, we suspect that the transition is second order. We have looked for a confirmation of this suspicion by applying the mean field approximation to the model (1). We introduce the mean field approximation in the standard way (Brézin *et al* 1976) to obtain a lower bound for the partition function

$$Z=\mathrm{Tr}_s\,\mathrm{e}^{A[s]}\geq \mathrm{Tr}_s\exp\left(B\sum_i s_i\right)\exp\left(\left\langle A[s]-B\sum_i s_i\right\rangle_B\right)=Z_{\mathrm{MF}} \tag{12}$$

where B is the mean field and $\langle\ \rangle_B$ represents the mean over the statistical weight $\exp(B\Sigma_i s_i)$. Now let us write the action (1) in the form

$$A[s]=\sum_{i,j}s_iJ_{ij}s_j=\sum_{i,j}s_i\left(\frac{k}{2}\sum_{\mu=1}^{d}(\delta_{i,j+\mu}+\delta_{i,j-\mu})+\frac{\lambda}{N}\right)s_j. \tag{13}$$

We can derive, therefore, an upper bound for the free energy per site given by

$$\mathscr{F}\leq\mathscr{F}_{\mathrm{MF}}=B\,\mathrm{d}u/\mathrm{d}B-u-(\mathrm{d}u/\mathrm{d}B)^2\tilde{J} \tag{14}$$

where

$$u=u(B)=\ln\cosh B,\qquad \tilde{J}=N^{-1}\sum_{i,j}J_{ij}.$$

Introducing the mean field order parameter

$$\mathrm{d}u/\mathrm{d}B=\langle s\rangle_{\mathrm{MF}}\equiv m, \tag{15}$$

one obtains

$$\mathscr{F}_{\mathrm{MF}}=-m^2\tilde{J}+\tfrac{1}{2}(1+m)\ln(1+m)+\tfrac{1}{2}(1-m)\ln(1-m). \tag{16}$$

This expression has only one minimum in $m=0$ if $\tilde{J}<\frac{1}{2}$ and two symmetric minima if $\tilde{J}>\frac{1}{2}$. In our case, therefore, the critical line in the plane (k,λ) is $\tilde{J}_c\equiv dk_c+\lambda_c=\frac{1}{2}$. This straight line is shown in figure 1. It is easy to verify that there is no discontinuity around $\tilde{J}_c$ in the first derivative of $\mathscr{F}_{\mathrm{MF}}$ with respect to $\tilde{J}$, while a discontinuity is present in the second derivative. We conclude that the transitions are second order. This is in agreement with the numerical results obtained by the Migdal–Kadanoff method.

We want to stress that we have been able, at least for this model, to obtain a sensible critical behaviour of the free energy through a lower bound approximation.

Further work is in progress to analyse the case $\lambda < 0$, to extend the results to any dimension and to study the critical exponents. We want to note, finally, that the model (1) can be easily generalised to a Potts-like model (Potts 1952), and that in this case the role of the first-order phase transition would be interesting to analyse.

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